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Universal level dynamics of complex systems

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We study the evolution of the distribution of eigenvalues of a $N\times N$ matrix subject to a random perturbation drawn from (i) a generalized Gaussian ensemble and (ii) a non-Gaussian ensemble with a measure variable under the change of basis. It turns out that, in case (i), a redefiniton of the parameter governing the evolution leads to a Fokker-Planck equation similar to the one obtained when the perturbation is taken from a standard Gaussian ensemble (with invaraiant measure). This equivalence can therefore help us to obtain the correlations for various physically significant cases modeled by generalized Gaussian ensembles by using the already known correlations for standard Gaussian ensembles. For large N values, our results for both cases (i) and (ii) are similar to those obtained for the Wigner-Dyson gas as well as for the perturbation taken from a standard Gaussian ensemble. This seems to suggest the independence of evolution, in the thermodynamic limit, from the nature of perturbation involved as well as the initial conditions and therefore the universality of dynamics of the eigenvalues of complex systems. [S1063-651X(99)04404-9]

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I. INTRODUCTION

Many important physical properties of complex quantum systems can be studied by analyzing the statistical response of eigenvalues to an external perturbation [1,2]. It turns out that the evolution of the eigenvalues of the system as a function of the strength of the perturbation is universal but only after an appropriate normalization of the perturbing potential [1]. Various approches have beeen adopted for a better understanding of this phenomenon, e.g., nonlinear σ model [3], random matrix theory (RMT) [4], semiclassical techniques [5], etc. Among these, a special class of Gaussian ensembles of RMT have been used very successfully in modeling the short energy-range behavior of eigenvalues [4]. The probability density of a matrix in this ensemble depends only on the functions invariant under change of basis, e.g., trace of the matrix thus making it easier to calculate the eigenvalue distribution. However, it has been conjectured that the local statistical properties of a few eigenvalues of a random Hermitian matrix are independent of the details of the distribution of the matrix elements, apart from a few broad characterizations such as real symmetric versus complex Hermitian, etc. (Chap. 1 of Ref. [4]). Therefore, the success of RMT in modeling the energy-level behavior of complex systems should not be restricted only to the standard Gaussian ensembles.

In this analytical work, we study this conjecture by examining the dynamics of eigenvalues of a quantum Hamiltonian under a random perturbation belonging to a class of (i) a general Gaussian ensemble and (ii) a non-Gaussian ensemble with logarithm of the distribution given by a polynomial. In both cases, the probability density of a matrix is chosen to contain the basis-dependent functions and therefore is no longer invariant under the basis transformation. By integrating over all perturbations from such an ensemble, we show that, for large N, the probability density of eigenvalues evolves according to a Fokker-Planck (FP) equation similar to the one proposed by Dyson for the Wigner-Dyson gas [6]. The strength of the perturbation acts here as a timelike coordinate. For finite N, this analogy seems to survive only when the dynamics is considered as a function of the perturbation strength as well as the variances of the matrix elements of the perturbation.

The statistical response of eigenvalues to a changing perturbation was studied by Simons and co-workers [1] by using a different approach, namely, the nonlinear σ model. By considering a Hamiltonian $H=H_0+xV$ with V as the perturbation and x its strength, they explicitly calculate the autocorrelation function of the energy eigenvalues for disordered systems within the so-called Gaussian approximation (or zero mode aproximation). Other results pertain to the numerical study of perturbed quantum chaotic systems, such as the chaotic billiard with a varying magnetic flux [7]. As shown in their work, the universality manifests itself among the density correlators only after the normalization of eigenvalues λ_i by mean level spacing and that of perturbation by the root mean square ''velocity'' of eigenvalues $\langle(\partial\lambda_i/\partial x)^2\rangle$; the universality here implies the independence of the correlators in the bulk of a system from system-dependent features. By treating x as a timelike parameter, they also conjectured the asymptotic correspondence of the parametric level-density correlators of quantum chaotic systems with the corresponding time-dependent particle-density correlators for the ground-state dynamics of the Sutherland Hamiltonian [8]. The connection between the static correlators (for a given parameter value) and the ground-state correlators (for a given time) of the Sutherland system is already well known [1].

This correspondence was analytically shown in a recent work by Narayan and Shastry [9]. They studied the evolution of the distribution of eigenvalues of a $N\times N$ matrix H subject to a random perturbing matrix V taken from a standard Gaussian (SG) ensemble and showed that it leads to the FP

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equation postulated by Dyson. Within this model, they proved the equivalence between the space-time correlators of the ground-state dynamics of an integrable one-dimensional (1D) interacting many-body quantum system [the Calogero-Sutherland (CS) system] and the second order parametric correlations of H. This analogy follows because the Dyson FP equation is equivalent, under a Wick rotation, to the quantum mechanics of the Calogero model, which, in the thermodynamic limit, has bulk properties identical to those of the Sutherland system [9,10].

Our present work extends this analogy further to a more general model of Hamiltonians H. We proceed as follows. For the clarification of our ideas, we first study, in Sec. II, the distribution of eigenvalues $P(\mu,\tau)$ for an arbitrary initial condition H_0 and the perturbation V taken from a generalized Gaussian ensemble (later referred to as the GG case). We obtain a partial differential equation governing its evolution, which, after certain parametric redefinitions, turns out to be formally the same as the Fokker-Planck equation governing the evolution of the Wigner-Dyson (WD) gas. As the solution of FP equation as well as various correlations in the latter case are already known for many initial conditions, this helps us to obtain $P(\mu,\tau)$ as well as the correlations in the GG case, as given in Sec. III. Section IV deals with the evolution of eigenvalues under a more general perturbation. We are motivated to study this case because the claim about universality of the dynamics can only be made if the equivalence is proved, at least in the thermodynamic limit, for a V with distribution of a more general nature. We conclude in Sec. V, which is followed by an appendix containing the proofs of some of the results used in this paper.

II. GENERALIZED GAUSSIAN CASE

Our interest is in the evolution of eigenvalues of H under a random perturbation V with initial state H_0 as in Ref. [9]. Here the both H_0 and V are Hermitian and can be assumed to have same mean spacing without any loss of generality. To ensure the same mean spacing for H also, we need to modify the parametrization of our perturbation strength x [9–11] from $H(x)=H_0+xV$ to $H(x)=H_0\cos(\Omega x)+V\Omega^{-1}\sin(\Omega x)$ with $\Omega\propto 1/\sqrt{N}$. Since $\Omega\to 0$ for large N, the reparametrization of the perturbation strength is inconsequential for finite x. The distribution $\rho(V)$ of matrix V is still chosen to be a Gaussian, $\rho(V)=C\exp(-\Sigma_{k\leq l}\alpha_{kl}V_{kl}^2)$ with C as the normalization constant. So far we are in the same situation as in Ref. [9] but now we depart and allow the variances for diagonal and off-diagonal matrix elements to be arbitrary (later referred to as the GG case) unlike the SG case where variance of the diagonal ones was twice that of the off-diagonal ones.

As indicated in [9], the evolution of eigenvalues of H as a function of x for any perturbation V can be expressed as a set of first order differential equations. They turn out to be analogous to the equations of motion of a classically integrable system with $2N+\beta N(N-1)/2$ degrees of freedom (with β a symmetry-dependent parameter). In principle, the distribution of eigenvalues can be obtained from these equations and its evolution can be studied but the coupling between the equations makes such a method very difficult. Thus we adopt another approach, the one used in [10] for studying the symmetric Gaussian case. As shown in [9,10], the proper form of FP equation can be achieved only after a further reparametrization, namely, $\tau=-\Omega^{-2}\ln\cos(\Omega x)$ and by studying the evolution in terms of τ. This suggests to us the use of the same parametrization for our case also, which gives

$$H(\tau)=H_0fh+Vh\Omega^{-1} \tag{1}$$

with $h=\sqrt{1-e^{-2\Omega^2\tau}}$ and $fh=e^{-\Omega^2\tau}$.

Given an ensemble of the matrices V distributed with a probability density $\rho(V)$ in the Hermitian matrix space, let $P(\mu,\tau)$ be the probability of finding eigenvalues $\lambda_i[V]$ of H between μ_i and $\mu_i+d\mu_i$ at ''time'' τ for an arbitrarily chosen H_0, which can be expressed as follows:

$$P(\mu_i,\tau)=\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i[V])\rho(V)dV. \tag{2}$$

Note here that $P(\mu,\tau)$ is in fact the conditional probability $P(\mu,\tau|\mu_0)$ with μ_0 as the eigenvalue matrix of H_0 but for simplification we use the former notation. As the τ dependence of P in Eq. (2) enters only through λ_i, a derivative of P with respect to τ can be written as follows [10]:

$$\frac{\partial P(\mu_i,\tau)}{\partial\tau}=-\sum_{n=1}^{N}\int\prod_{i=1\neq n}^{N}\delta(\mu_i-\lambda_i)\times\frac{\partial\delta(\mu_n-\lambda_n)}{\partial\mu_n}\frac{\partial\lambda_n}{\partial\tau}\rho(V)dV. \tag{3}$$

The further treatment of the above equation depends on the symmetry classes of the matrices V and H, that is, whether they are real-symmetric or complex Hermitian.

In the real-symmetric case, $\partial\lambda_n/\partial\tau$ can be expressed in terms of the matrix elements of the orthogonal matrix O, which diagonalizes H, $H=O^T\Lambda O$ with Λ as the eigenvalue matrix (Appendix A). Using this in Eq. (3) leads to the following form with $g_{kl}=(1+\delta_{kl})$,

$$\frac{\partial P(\mu_i,\tau)}{\partial\tau}=\Omega^2\sum_n\frac{\partial}{\partial\mu_n}(\mu_nP)-2h^{-1}\Omega\sum_n\frac{\partial}{\partial\mu_n}\times\int\prod_i\delta(\mu_i-\lambda_i)\sum_{k\leq l}O_{nk}O_{nl}\frac{V_{lk}}{g_{kl}}\rho(V)dV. \tag{4}$$

An application of the partial integration on the second term in the right of the above equation then gives

$$\frac{\partial P(\mu_i,\tau)}{\partial\tau}=\Omega^2\sum_n\frac{\partial}{\partial\mu_n}(\mu_nP)-h^{-1}\Omega\sum_n\frac{\partial}{\partial\mu_n}\times\int\sum_{k\leq l}\frac{1}{\alpha_{kl}g_{kl}}\frac{\partial}{\partial V_{kl}}\left[\prod_i\delta(\mu_i-\lambda_i)O_{nk}O_{nl}\right]\times\rho(V)dV. \tag{5}$$

For simplification let us assume the same variance for all the diagonal matrix elements such that $g_{kl}\alpha_{kl}=\alpha$ for $k=l$. Now, by expressing $\Sigma_{k\leq l}(1/\alpha_{kl}g_{kl})\cdots=1/\alpha\,\Sigma_{k\leq l}\cdots$

$-\Sigma_{k<l}[(1/\alpha)-(1/g_{kl}\alpha_{kl})]$ and with the help of real-symmetric analog of relations (A3)–(A8) as well as the partial integration technique, we can reduce Eq. (5) as follows:

$$\frac{\partial P(\mu_i,\tau)}{\partial \tau}=\Omega^2\sum_n \frac{\partial}{\partial \mu_n}(\mu_n P)+\frac{\Omega}{h\alpha}\sum_n \frac{\partial}{\partial \mu_n}\sum_m \frac{\partial}{\partial \mu_m} \times \int \prod_i \delta(\mu_i-\lambda_i)\sum_{k\leqslant l}\left[\frac{\partial \lambda_m}{\partial V_{kl}}O_{nk}O_{nl}\right]\rho(V)dV + \frac{1}{\alpha}\sum_{n,m,n\neq m}\frac{\partial}{\partial \mu_n}\int \prod_i \delta(\mu_i-\lambda_i)\frac{\rho(V)}{\lambda_m-\lambda_n}dV - h^{-1}\Omega\sum_{k<l}(g_{kl}^{-1}\alpha_{kl}^{-1}-\alpha^{-1})\sum_n \frac{\partial}{\partial \mu_n} \times \int\left[\frac{\partial}{\partial V_{kl}}\left(\prod_i \delta(\mu_i-\lambda_i)O_{nk}O_{nl}\right)\right]\rho(V)dV. \tag{6}$$

By applying the orthogonality relation of matrix O, Eq. (6) can now be rewritten as follows:

$$\frac{\partial P(\mu_i,\tau)}{\partial \tau}=\Omega^2\sum_n \frac{\partial}{\partial \mu_n}(\mu_n P) + \frac{1}{\alpha}\sum_n \frac{\partial}{\partial \mu_n}\left[\frac{\partial}{\partial \mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]P-F, \tag{7}$$

where $\beta=1$ and

$$F=h^{-2}\Omega^2\sum_{k<l}(g_{kl}^{-1}\alpha_{kl}^{-1}-\alpha^{-1})\int \prod_i \delta(\mu_i-\lambda_i) \times[\alpha_{kl}-2\alpha_{kl}^2V_{kl}^2]\rho(V)dV. \tag{8}$$

Following similar steps and the appropriate relations given in the Appendix, one obtains the same equation as in the complex Hermitian case (with $\beta=2$). Further analysis of Eq. (7) depends on the relative values of variances for the off-diagonal matrix elements. For a clear understanding, let us first consider a case where all the off-diagonals have the same variances, such that $g_{kl}\alpha_{kl}=\alpha'$ (for $k\neq l$). F can then be expressed in terms of the derivative of P with respect to $y\equiv\alpha'/\alpha$:

$$F=2h^{-2}\Omega^2(1-y)y\frac{\partial P}{\partial y}. \tag{9}$$

Here we have used an idea quite important to our analysis that the variances of the matrix elements of V can also vary and therefore eigenvalues evolve not only as a function of the perturbation strength but of the variances too. Note here that F can also be expressed as an extra drift term ($F=\Sigma_n\{[\partial R(\mu)]/\partial\mu_n\}$), with $R=h^{-1}\Omega\Sigma_{k\leqslant l}(g_{kl}^{-1}\alpha_{kl}^{-1}-\alpha^{-1})\int\{(\partial/\partial V_{kl})[\Pi_i\delta(\mu_i-\lambda_i)O_{nk}O_{nl}]\}\rho(V)dV$ and it can be of interest to study the effect of this extra drift on the dynamics. For example, for R nonlinear in P, the motion may lose its random behavior, showing some sort of periodicity. But in this paper, we restrict ourselves just to exploring the universality of the dynamics, that is, to verifying that P satisfies a FP equation similar to the one proposed by Dyson.

Equation (7) contains the derivatives of P with respect to two parameters, namely, τ and y. In order to prove that the statistical quantities in the GG case evolve in a similar way as in the SG case, one should be able to reduce the two-parameter dependence of Eq. (7) to a single-parameter dependence. We assume that it is possible along a curve parametrized by $\phi,\tau=\tau(\phi),y=y(\phi)$ such that

$$\frac{\partial}{\partial \phi}=\frac{\partial \tau}{\partial \phi}\frac{\partial}{\partial \tau}+\frac{\partial y}{\partial \phi}\frac{\partial}{\partial y}. \tag{10}$$

A comparison of Eq. (10) with Eq. (7) then leads us to the conditions for existence of such a curve, namely, $\partial\tau/\partial\phi=1$ and $\partial y/\partial\phi=[2\Omega^2y(1-y)]/h^2$. By solving these equations, one obtains the following curve in the (y,τ) space:

$$y=\frac{e^{2\Omega^2\tau}-1}{e^{2\Omega^2\tau}+y_0} \quad \text{and} \quad \tau=\phi+\tau_0, \tag{11}$$

where τ_0 and y_0 are arbitrary constants. Note that the conditions mentioned above do not impose any constraints on these constants except that the curve should remain on the positive side of the upper half of y-τ plane. The extraction of τ_0 from Eq. (11) leads to the following form of ϕ:

$$\phi=\tau-\frac{1}{2\Omega^2}\ln\left[1+d\frac{|y-1|}{y}(e^{2\Omega^2\tau}-1)\right], \tag{12}$$

where $d=y_0/(y_0-1)$ and can be chosen as unity (since the corresponding value of ϕ still satisfies the required conditions). Thus, for $y=\alpha'/\alpha$, Eq. (11) represents a set of curves along which Eq. (7) adopts the form of a FP equation:

$$\frac{\partial P(\mu_i,\phi)}{\partial \phi}=\Omega^2\sum_n \frac{\partial}{\partial \mu_n}(\mu_n P) + \frac{1}{\alpha}\sum_n \frac{\partial}{\partial \mu_n}\left[\frac{\partial}{\partial \mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]P, \tag{13}$$

where the steady state is achieved for $\phi\to\infty$ which corresponds to $\tau\to\infty$ and $\alpha\to\alpha'$; the steady-state solution is given by $\Pi_{i<j}|\mu_i-\mu_j|^\beta e^{-(\alpha/2)\Omega^2\Sigma_k\mu_k^2}$. Note that only $\tau\to\infty$ (with $\partial P/\partial\tau=0$) no longer represents the steady state as in the SG or WD case but represents a transition state with $\phi(y)=-(1/2\Omega^2)\ln[d(|y-1|/y)]$.

The steady-state limit of P at $(y,\tau)=(1,\infty)$ seems to indicate that if the initial perturbation is such that the width ($\propto 1/\alpha$) of the eigenfunctions of H is smaller than the overlapping between them ($\propto 1/\alpha'$), they have a tendency to overcome this difference and extend over all of the available Hilbert space. This also seems to suggest that a complicated interaction giving rise to larger transition probabilities between energy levels as compared to the probability of staying in the same level can only be transient in nature. Given the freedom to change, it rearranges itself in such a way as to equalize these probabilities.

Equation (13) is formally the same as the FP equation governing the Brownian motion of particles, in time, of the Wigner-Dyson gas. However, in the former case, the transition parameter ϕ is a function of both perturbation strength and the relative variances of matrix elements. In the limit $N\rightarrow\infty$, this two-parameter dependence of ϕ reduces to one ($\lim N\rightarrow\infty$, $\phi=\tau$). This can also be seen directly from Eqs. (7) and (8) because in the limit $N\rightarrow\infty$, $\Omega\rightarrow 0$ and $H\rightarrow H_0+\tau\sqrt{2}V$, thus removing any Ω dependence of λ_i and making $F\rightarrow 0$. A same result for SG-type perturbation has already been obtained [9]. The analogous behavior in both GG and SG case can be attributed to the negligible effect of the distribution of N-diagonal matrix elements on the dynamics in the presence of $N(N-1)/2$ off-diagonal ones. This indicates that, in the thermodynamic limit, differences in the variances do not affect the evolution of eigenvalues.

Again, in the $N\rightarrow\infty$ limit, all three distributions, namely, SG, GG, and WD, evolve in the same way for arbitrary initial conditions. Thus, all moments of the eigenvalues in SG and GG cases at any value of the transition parameter will be equal to the corresponding moments of particle positions of a WD gas undergoing Brownian motion due to the presence of thermal noise. However, as discussed in [9], this does not imply a Brownian motion of eigenvalues for the first two cases, the reason being the difference of sources randomizing the motion of eigenvalues. For SG and GG cases, the source is the matrix V, acting like a quenched disorder while, for the WD gas, the thermal noise gives rise to an annealed randomness. Although these different origins of randomness do not affect the static (for one-parameter value) or the second-order parametric correlations, as shown in Ref. [9], multiple parametric correlations higher than the second order are different for the perturbed systems and WD gas. However, the origin of randomness being similar for the SG and GG cases, all the conclusions obtained for the correlations for the former [9] are also valid for the latter in the thermodynamic limit.

It is interesting to note that Eq. (4) can also be written in the following form [by first using the real-symmetric analog of Eq. (A3) in Eq. (4) and then taking the derivative with respect to μ_n inside the integral]:

$$\frac{\partial P(\mu_i,\tau)}{\partial\tau}=\Omega^2\sum_n \frac{\partial}{\partial\mu_n}(\mu_n P)+h^{-2}\Omega^2\sum_{k\leqslant l}\times\int \frac{\partial\prod_i \delta(\mu_i-\lambda_i)}{\partial V_{kl}}V_{kl}\rho(V)dV. \quad (14)$$

Now an application of the partial integration to the second term on the right-hand side of this equation gives

$$\int \frac{\partial\prod_i \delta(\mu_i-\lambda_i)}{\partial V_{kl}}V_{kl}\rho(V)dV =\int\prod_i \delta(\mu_i-\lambda_i)[1+2\alpha_{kl}V_{kl}^2]\rho(V)dV. \quad (15)$$

By expressing this in terms of the derivatives of P with respect to α,α' as before, we obtain a drift equation (without a diffusion term as well as the repulsive pairwise potential)

$$\frac{\partial P(\mu_i,\tau)}{\partial\phi_1}=\Omega^2\sum_n \frac{\partial}{\partial\mu_n}(\mu_n P), \quad (16)$$

where ϕ_1 describes a curve in a three-dimensional (y,α,α') space such that $\partial P/\partial\phi_1=(\partial P/\partial\tau)+(2\Omega^2/h^2)[\alpha'(\partial P/\partial\alpha')+\alpha(\partial P/\partial\alpha)]$. The above possibility of reduction of the FP equation in a pure drift form, just by going to a higher parametric space, implies the total blindness of eigenvalues to any mutually repulsive potential or locally fluctuating force in this space. Note that it is already well known that a repulsion between the eigenvalues vanishes if the number of parameters of the Hamiltonian undergoing variation is more than one in the real-symmetric case (two for the complex-Hermitian case) [11]. However, the same situation seems to prevail even when the parameters undergoing variation are the ones governing the distribution of matrix elements of Hamiltonians.

So far, we have considered the case where all the off-diagonal matrix elements V_{ij} have the same variance although different from that of the diagonal. But in many physical situations, e.g., in disordered systems, there occurs localization of eigenfunctions that may give rise to off-diagonal elements with different variances. One such situation most often encountered is where V_{ij}'s decay exponentially with respect to the distance from the diagonal. It can be modeled by taking different variances for the different off-diagonals, that is, $g_{kl}\alpha_{kl}=\alpha_r$ $(r=|k-l|)$. F in such a case is given by, with $y_r\equiv\alpha_r/\alpha$,

$$F=2h^{-2}\Omega^2\sum_{r=1}^{N}(1-y_r)y_r\frac{\partial P}{\partial y_r}. \quad (17)$$

We assume that it is possible along a curve parametrized by ϕ, $\tau=\tau(\phi),Y=Y(y_1,\ldots,y_N)$ such that

$$\frac{\partial}{\partial\phi}=\frac{\partial\tau}{\partial\phi}\frac{\partial}{\partial\tau}+\frac{\partial Y}{\partial\phi}\frac{\partial}{\partial Y} \quad (18)$$

with $\partial/\partial Y=\Sigma_r(\partial y_r/\partial Y)(\partial/\partial y_r)$. Reasoning again as before one finds that the universality of evolution of the probability density still survives but only by going to a higher (N-dimensional) parametric space $(\tau,y_1,\ldots,y_N)$, along a set of curves parametrized by ϕ and given by the conditions $(\partial\tau/\partial\phi)=1$; $(\partial Y/\partial\phi)=2h^{-2}\Omega^2$ with $(\partial y_r/\partial Y)=(1-y_r)y_r$. Again, ϕ can be obtained by solving these equations:

$$\phi=\tau-\frac{1}{2\Omega^2}\ln\left[1+d\prod_{r=1}^{N}\left(\frac{|y_r-1|}{y_r}\right)(e^{2\Omega^2\tau}-1)\right]. \quad (19)$$

Note here that Π_r contains the contribution only from those y_r's for which $y_r-1\neq 0$.

One can also consider a physical situation when all the off-diagonal matrix elements have different probability laws $g_{kl}\alpha_{kl}=\alpha$(for $k=l$) and $\alpha_{kl}\neq\alpha_{ij}$ if $(kl\neq ij)$. In this case, F now turns out to be as follows with $y_{kl}\equiv\alpha_{kl}/\alpha$:

$$F=2h^{-2}\Omega^2\sum_{k<l}(1-y_{kl})y_{kl}\frac{\partial P}{\partial y_{kl}}. \tag{20}$$

Now one has to go to $\{[N(N-1)]/2\}+1$-dimensional parametric space to recover the FP equation similar to that of the Wigner-Dyson gas. The curves in this case are again parametrized by ϕ, which is still related to τ and Y by Eq. (18) but now $Y\equiv Y(\{y_{kl}\})$ is such that $\partial/\partial Y=\Sigma_{k<l}(\partial y_{kl}/\partial Y)(\partial/\partial y_{kl})$ with $\partial y_{kl}/\partial Y=(1-y_{kl})y_{kl}$; the two other derivatives of τ and Y with respect to ϕ remain the same as in the above case. Proceeding exactly as before, ϕ in this case can be shown to be the following:

$$\phi=\tau-\frac{1}{2\Omega^2}\ln\left[1+d\prod_{k<l}^{N}\left(\frac{|y_{kl}-1|}{y_{kl}}\right)(e^{2\Omega^2\tau}-1)\right]. \tag{21}$$

Again, as before, $\Pi_{k<l}$ contains contributions from all $y_{kl}\neq 1$. We have discussed here very few of the various possibilities one may encounter in complex systems. As indicated by the cases discussed above, the desired form of Eq. (13) can only be obtained by an appropriate partitioning of the sums [e.g., as done near Eq. (6)], which leads to a separation of the contributions from terms with unequal variances from those with equal variances. The latter gives the required drift and diffusion terms while the former is absorbed in a parametric derivative (given by F). Using the same technique, F can be written for other cases too.

For the SG case, ϕ appearing in the correlators of the type $\langle d(E,\phi),d(E,0)\rangle$ involves variation of just one parameter, for the CG case two parameters or more. However, in both cases, the expressions of second-order correlators is the same due to the analogous form of the FP equation. Our study therefore suggests that the second-order correlators involving variation of more than one parameter can always be expressed as those involving just one effective parameter. For example, the correlators belonging to the RM models of both localized as well as delocalized cases can be expressed in the same form by using parameter ϕ although the definition of ϕ is different in the two cases.

Our study also suggests that the nature of the dynamics of the eigenvalues is very sensitive to the number of parameters undergoing a change. For finite N, the motion may appear as pure drift, suggesting no interaction between eigenvalues and a total absence of local fluctuating and frictional forces if variances of all the matrix elements are allowed to vary independently. It seems to be Brownian as a function of just the relative variations of the variance. Under variation of only the perturbation parameter, the presence of an extra driftlike term makes the motion non-Brownian. But, the contribution of this term being zero in the $N\to\infty$ limit, it recovers its Brownian nature. It can be of interest to study the effect of this extra drift on the dynamics.

Although the study presented here deals with the energy-level dynamics of random matrix-models of Hermitian operators, e.g., the Hamiltonian of complex systems, the results are also valid for the level dynamics of unitary operators U, e.g., the time-evolution operator being perturbed by a random matrix V taken from the GG ensembles, $\{U(\tau)=U_0\exp[i\sqrt{\tau}V]\approx U_0[1+i\sqrt{\tau}V]$ for small τ values$\}$. The required FP equation in this case turns out to be the following:

$$\frac{\partial P(\mu_i,\phi)}{\partial\phi}=\frac{1}{\alpha}\sum_n\frac{\partial}{\partial\mu_n}\left[\frac{\partial}{\partial\mu_n}+\frac{\beta}{2}\sum_{m\neq n}\cot\left(\frac{\mu_m-\mu_n}{2}\right)\right]P, \tag{22}$$

with ϕ defined as before. To prove the equivalence for U a symmetric unitary matrix $(\beta=1)$, one has to follow steps similar to those in the real-symmetric case of Hermitian matrices, as both are invariant under orthogonal transformation. For a general unitary U $(\beta=2)$, the steps are similar to the complex Hermitian case. As is obious from Eq. (22), this is similar to the FP equation obtained if V belongs to the SG ensembles [12].

III. CALCULATION OF $P(\mu,\tau)$ AND CORRELATIONS

In the preceding section, we obtained a FP equation governing the evolution of probability $P(\mu,\tau)$ for an arbitrarily given $H(\phi_0)$ (the conditional probability). Here ϕ_0 corresponds to the case $\tau=0$ and and $H(\phi_0)$ can be obtained by using the relation $H(\phi)=H(\phi_0)e^{-\Omega^2(\phi-\phi_0)}+\hat{V}\sqrt{1-e^{-2\Omega^2(\phi-\phi_0)}}$, substituting ϕ in terms of τ and y there and then comparing it with the expression for the Hamiltonian $H(\tau)$ given by Eq. (1). Here $\hat{V}$ is a matrix with an invariant probability distribution $[\rho(\hat{V})\propto e^{-(\alpha/2)\mathrm{Tr}\,\hat{V}^2}]$ and $H(\phi)$ can be written in the above form (the ''SG form'') since it gives the correct evolution equation for the probability distribution.

The formal similarity of the FP equation to that of the WD gas case as well as the SG case makes it easier to obtain $P(\mu,\tau)$ at least for $\beta=2$, since the solution of Eq. (13) [and (22)] for this β value is already known [4,12]. The $P(\mu,\phi)$ [and therefore $P(\mu,\tau)$] can also be calculated directly from the Hamiltonian $H(\phi)$ by using the sum of the matrices technique (that is, by evaluating a two-matrix integral) as for the SG case [4] because now both of the component matrices have invariant distribution, unlike Eq. (1). For completeness, we give here a few of the steps, used in solving the SG case, for our case; for details, refer to [12].

As can readily be checked, both Eqs. (13) and (22) can also be written as follows (with $\alpha=1$ for simplification):

$$\frac{\partial P(\mu_i,\phi)}{\partial\phi}=\sum_n\frac{\partial}{\partial\mu_n}|Q_N|^\beta\frac{\partial}{\partial\mu_n}\frac{P}{|Q_N|^\beta}, \tag{23}$$

where $|Q_N|^\beta=|\Delta(\mu)|^\beta e^{-\Omega^2\Sigma_k\mu_k^2}$ with $\Delta(\mu)=\Pi_{i<j}(\mu_i-\mu_j)$ for the Hermitian case and $=\Pi_{j<k}\sin[(\mu_j-\mu_k)/2]$ for the unitary case. The transformation $\Psi=P/|Q_N|^{\beta/2}$ allows us to cast Eq. (23) in the suggestive form

$$\frac{\partial\Psi}{\partial\phi}=-\hat{H}\Psi, \tag{24}$$

where, for the Hermitian case, the ''Hamiltonian'' $\hat{H}$ turns out to be the Calogero-Moser (CM) Hamiltonian

$$\hat{H}=\sum_i\frac{\partial^2}{\partial\mu_i^2}-\frac{1}{2}\sum_{i<j}\frac{\beta(\beta-2)}{(\mu_i-\mu_j)^2}+\frac{\Omega^4}{4}\sum_i\mu_i^2. \tag{25}$$

Similarly, for the unitary case, $\hat{H}$ is the Calogero-Sutherland Hamiltonian

$$\hat{H} = -\sum_i \frac{\partial^2}{\partial \mu_i^2} + \frac{\beta(\beta-2)}{16} \sum_{i \neq j} \operatorname{cosec}^2(\mu_i - \mu_j) - \frac{\beta^2}{48} N(N^2-1). \quad (26)$$

With the parabolic-confining potential (or periodic boundary conditions) and under the requirement (to take into account the singularity in H) that the solutions vanish as $|\mu_i - \mu_j|^{\beta/2}$ when μ_i and μ_j are close to each other, $\hat{H}$, in both Eqs. (25) and (26), has well-defined (completely symmetric) eigenstates ζ_k and eigenvalues λ_k. This allows us to express the "state" Ψ and therefore $P(\mu,\phi|H_0)$ as a sum over eigenvalues and eigenfunctions of $\hat{H}$,

$$P(\mu,\phi|H(\phi_0)) = \left|\frac{Q_N(\mu)}{Q_N(\mu_0)}\right|^{\beta/2} \times \sum_{k>0} \exp[-\lambda_k(\phi-\phi_0)]\zeta_k(\mu)\zeta_k^*(\mu_0), \quad (27)$$

where $\mu_0 \equiv (\mu_{01}, \mu_{02}, \ldots, \mu_{0N})$ are the eigenvalues of $H(\phi_0)$. The joint probability distribution $P(\mu,\phi)$ can then be obtained by integrating over all initial conditions,

$$P(\mu,\phi) = \int P(\mu,\phi|\mu_0,\phi_0) P(\mu_0,\phi_0) d\mu_0, \quad (28)$$

which further leads to the correlations $R_n(\mu_1,\ldots,\mu_n;\phi) = \{N!/[(N-n)!]\int d\mu_{n+1}\ldots d\mu_N P(\mu_1,\ldots,\mu_N;\phi)$ using standard techniques [12,13]. In fact, a direct integration of the FP equation (13) leads to the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchic relations among the unfolded correlators $R_n(r_1,\ldots,r_n;\Lambda) = \lim N \to \infty [R_n(\mu_1,\ldots,\mu_n;\phi)]/R_1(\mu_1;\phi)\ldots R_1(\mu_n;\phi)$ with $r = \int^r R_1(\mu;\phi)d\phi$ and $\Lambda = \phi/D^2$ (D being the mean level spacing) [12,13],

$$\frac{\partial R_n}{\partial \Lambda} = \sum_j \frac{\partial^2 R_n}{\partial r_j^2} - \beta \sum_{j\neq k} \frac{\partial}{\partial r_j}\left(\frac{R_n}{r_j - r_k}\right) - \beta \sum_j \frac{\partial}{\partial r_j} \int_{-\infty}^{\infty} \frac{R_{n+1}}{r_j - r_k}. \quad (29)$$

For $n=2$ and small values of r, the integral term makes a negligible contribution, thus leading to the following approximated closed-form equation for R_2:

$$\frac{\partial R_2}{\partial \Lambda} = 2\frac{\partial^2 R_2}{\partial r^2} - 2\beta \frac{\partial}{\partial r}\frac{R_2}{r}. \quad (30)$$

Equations (27) and (28) represent the formal solutions of Eq. (13) [or Eq. (22)]. To proceed further, one needs to know the eigenvalues and eigenfunctions of $\hat{H}$ so as to express P in a compact form, but these are explicitly known only for the $\beta=2$ case [8]. This is because for $\beta=2$ the interaction term in Eqs. (25) and (26) drops out and P can explicitly be obtained. For the CM model [Eq. (25)], it is given as follows [4]:

$$P(\mu,\phi|H(\phi_0);\beta=2) \propto \left|\frac{\Delta(\mu)}{\Delta(\mu_0)}\right| \det[f_m(\mu_i - \mu_{0j};\phi-\phi_0)]_{i,j=1\ldots N}, \quad (31)$$

with $f_m(x-y;t) = \exp\{-[(xe^{\Omega^2 t} - y)/(e^{2\Omega^2 t}-1)]\}$, and, for the CS model [12],

$$P(\mu,\phi|H(\phi_0);\beta=2) = \frac{1}{N!}\left|\frac{\Delta(\mu)}{\Delta(\mu_0)}\right| \det[f_s(\mu_i - \mu_{0j};\phi-\phi_0)]_{i,j=1\ldots N}, \quad (32)$$

where $f_s(x) = 1/2\pi \sum_{k=-\infty}^{\infty} \exp[-k^2\phi + ikx]$.

Since we already know $P(\mu,\phi)$ and various correlations for the WD case (for which $\phi=\tau$) starting from various initial conditions [12,13], one can obtain these measures for the GG case just by replacing ϕ by its appropriate relationship with τ and the variances of the perturbation-matrix elements. For example, for the case $H = V\Omega^{-1}$ [which corresponds to $H(\tau\to\infty)$ in Eq. (1)] with different variances for the different diagonals of V [Eq. (17)], $\phi = -(2\Omega^2)^{-1}\ln\{\Pi_{r=1}^N(|y_r - 1|)/y_r]\}$ [with d chosen to be unity for the same reason as in Eq. (12)]. Also note that in this case $\partial P/\partial\tau = 0$. The initial condition $\phi=0$ now corresponds to all $y_r \to \infty$, thus implying the existence of only the diagonal elements of V and a Poisson distribution for $P(\mu_0,\phi=0)$ ($\propto e^{-\Omega^2\alpha\Sigma_i V_{ii}^2}$ with $V_{ii} = \mu_{0i}$). As $\phi\to\infty$ when $y_r \to 1$ for all r, the equilibrium distribution is therefore given by the standard Gaussian ensembles. This case thus corresponds to the Poisson → GE transition in the standard Gaussian ensembles [12,13] with ϕ now as a transition parameter (replacing τ, the perturbation strength) with intermediate ensembles representing the cases for any choice of y_r's. The two-point correlations for this transition for V belonging to the SG ensemble (complex-Hermitian) have already been obtained [13]. It should be noted that, as for the SG ensembles, ϕ must be rescaled to see the smooth transition [14].

IV. POLYNOMIAL CASE

In Sec. II, we showed the equivalence of particle dynamics of the Wigner-Dyson gas with the evolution of the eigenvalues of a Hamiltonian under a perturbation V taken from generalized Gaussian ensembles. However, the claim about universality of the dynamics and level-density correlators can only be made if the equivalence is proved, at least in the thermodynamic limit, for a V with distribution of a more general nature. In this section, we attempt to verify the claim by taking $\rho(V)$ as the exponential of the polynomial form of V, $\rho(V) = C\exp[-\Sigma_{k\leq l} Q(V_{kl})]$ with $Q(x) = \Sigma_{r=1}^M \gamma_{kl}(r)x^{2r}$ (a polynomial of x with degree $2M$), C as the normalization constant (referred to as polynomial case I later on) and variances for the diagonal and off-diagonal matrix elements chosen to be arbitrary. Note that the universality of the correlators for the case $H=V$, with V distributed as in this case, has already been studied in Refs. [15,16] (which corresponds to the steady-state limit of the study given here).

To obtain a FP equation, we now need the following equality, which can be proved by using integration by parts:

$$\int f[V]V_{kl}\rho(V)dV_{kl} = \frac{1}{2\gamma_{kl}(1)}\int \frac{\partial f[V]}{\partial V_{kl}}\rho(V)dV - \sum_{r=2}^{k} r\frac{\gamma_{kl}(r)}{\gamma_{kl}(1)}\int V_{kl}^{(2r-1)}f[V]\rho(V)dV. \quad (33)$$

By using the real symmetric analog of Eq. (A1) in Eq. (3), followed by the above equality, we get

$$\frac{\partial P}{\partial \tau} = \Omega^2\sum_n \frac{\partial}{\partial \mu_n}(\mu_n P) - h^{-1}\Omega\sum_n \frac{\partial}{\partial \mu_n}\int \sum_{k\leqslant l}\frac{1}{\gamma_{kl}(1)g_{kl}}\frac{\partial}{\partial V_{kl}} \times\left[\prod_i \delta(\mu_i-\lambda_i)O_{nk}O_{nl}\right]\rho(V)dV + 2h^{-1}\Omega\sum_{r=2}^{M} r\sum_n \frac{\partial}{\partial \mu_n}\int\prod_i \delta(\mu_i-\lambda_i) \times\left[\sum_{k\leqslant l}\frac{\gamma_{kl}(r)}{\gamma_{kl}(1)}\frac{V_{kl}^{2r-1}}{g_{kl}}O_{nk}O_{nl}\right]\rho(V)dV. \quad (34)$$

Now for simplification let us consider a case where $g_{kl}\gamma_{kl}(r)=\gamma(r)$ if $(k=l)$ and $g_{kl}\gamma_{kl}(r)=\gamma'(r)$ if $(k\neq l)$.

Using the tools given in Appendix A and by an extensive use of the integration by parts while dealing with difficult integrals, the first term appearing in the above equation can now be reduced to the same form as in the GG case. This results in the following:

$$\frac{\partial P(\{\mu_i\},\tau)}{\partial \tau} = \Omega^2\sum_n \frac{\partial}{\partial \mu_n}(\mu_n P) + \frac{1}{\gamma'(1)}\sum_n \frac{\partial}{\partial \mu_n} \times\left[\frac{\partial}{\partial \mu_n}+\sum_{m\neq n}\frac{\beta}{\lambda_m-\lambda_n}\right]P - Z, \quad (35)$$

where $\beta=1$ and

$$Z=\frac{\Omega^2}{h^2}\left(\frac{1}{\gamma(1)}-\frac{1}{\gamma'(1)}\right)F-\frac{\Omega^2}{h^2}\sum_{r=2}^{k} r \times\left[\frac{\gamma'(r)}{\gamma'(1)}G_{r1}+\frac{\gamma(r)}{\gamma(1)}G_{r2}\right] \quad (36)$$

with F, G_{r1}, and G_{r2} given as follows:

$$F=\frac{1}{2}\sum_n \frac{\partial}{\partial \mu_n}\int\left[\sum_k \frac{\partial}{\partial V_{kk}}\left(\prod_i \delta(\mu_i-\lambda_i)\frac{\partial \lambda_n}{\partial V_{kk}}g_{kk}\right)\right] \times\rho(V)dV, \quad (37)$$

$$G_{r1}=\sum_n \frac{\partial}{\partial \mu_n}\int\prod_i \delta(\mu_i-\lambda_i)\left[\sum_{k<l}\frac{\partial \lambda_n}{\partial V_{kl}}V_{kl}^{2r-1}\right]\rho(V)dV, \quad (38)$$

$$G_{r2}=\sum_n \frac{\partial}{\partial \mu_n}\int\prod_i \delta(\mu_i-\lambda_i)\left[\sum_k \frac{\partial \lambda_n}{\partial V_{kk}}V_{kk}^{2r-1}\right]\rho(V)dV. \quad (39)$$

We apply partial integration to F and rewrite it as follows:

$$F=\gamma(1)\sum_k \int\prod_i \delta(\mu_i-\lambda_i)\left[1-\gamma(1)V_{kk}^2 - \sum_{r=2}^{M} r\gamma(r)V_{kk}^{2r}\right]\rho(V)dV+\sum_{r=2}^{k} r\gamma(r)G_{r2}. \quad (40)$$

Similarly, G_{r1} and G_{r2} can be rewritten as follows:

$$G_{r1}=\sum_{k<l}\int\prod_i \delta(\mu_i-\lambda_i)\left[(2r-1)V_{kl}^{2r-2}-2\gamma'(1)V_{kl}^{2r} - 2\sum_{s=2}^{M} s\gamma'(s)V_{kl}^{2r+2s-2}\right]\rho(V)dV, \quad (41)$$

$$G_{r2}=\sum_k \int\prod_i \delta(\mu_i-\lambda_i)\left[(2r-1)V_{kk}^{2r-2}-\gamma(1)V_{kk}^{2r} - \sum_{s=2}^{M} s\gamma(s)V_{kk}^{2r+2s-2}\right]\rho(V)dV. \quad (42)$$

For the complex Hermitian case also, one obtains an equation similar to Eq. (35) with $\beta=2$ and Z given by Eq. (36). Note that all the terms appearing in the expressions of F, G_{r1}, and G_{r2} are of the type $\int\prod_i(\mu_i-\lambda_i)V^j\rho(V)dV$, and, as done in Sec. II for the GG case, they can easily be rewritten in terms of the derivatives with respect to $\gamma(r)$'s if $j\leqslant 2M$. It is difficult to do so for terms with $j>2M$; they probably could be expressed as higher-order derivatives with respect to more than one variance parameter (this is the case at least for $N=2$). However, as physically significant RM models of complex systems generally correspond to the limit $N\rightarrow\infty$, it would be sufficient, for our purpose, to study the behavior of the terms in this limit. As is obvious from Eqs. (40)–(42), the N or Ω dependence of F, G_{r1}, and G_{r2} is due to the presence of λ_i's as well as the coefficients $\gamma(r)$, $r=1\rightarrow M$. Let us consider the case where both $\gamma(r)$ and $\gamma(r')$, where $r=1\rightarrow M$, are independent of N and all the matrix elements of V are distributed with zero mean which implies the N independence of $\rho(V)$. Thus the N dependence of F, G_{r1}, and G_{r2} in this case is of the same order as appears in P, which can clearly be seen by using the equality $\delta(\lambda^{-1}[\mu]-V)=\delta(\mu-\lambda[V])\det|\partial\lambda/\partial V|$ and rewriting these integrals in terms of the function $\lambda^{-1}(\mu)$. Equation (36) therefore implies that the contribution from Z to Eq. (35) is negligible (as compared to the diffusion term and the drift term due to mutual repulsion) in the limit $N\rightarrow\infty$. Similarly for the GG case, Z vanishes for $N\rightarrow\infty$; Z here can be obtained by substituting $\gamma(r)=0$, $\gamma'(r)=0$, for $r>1$, in Eq. (13). Further, for the case where distribution, although basis-independent, contains a non-Gaussian term, that is, for $\rho(V)\propto\exp[-\alpha\,\mathrm{Tr}(V^2)-\Sigma_{r=2}^{M}\gamma(r)\mathrm{Tr}(V^{2r})]$ (polynomial case II), the Z can be shown to be the following:

$$Z=\frac{2\gamma\Omega^2}{h^2}\sum_{r=2}^{k} r\gamma(r),$$

$$\int \prod_i \delta(\mu_i-\lambda_i)\left[\sum_{k,l} g_{k,l}\frac{\partial(V^{2r-1})_{kl}}{\partial V_{kl}}-4\alpha\,\mathrm{Tr}(V^{2r})-4\sum_{s=2}^{k}\mathrm{Tr}(V^{2r+2s-2})\right]\rho(V)dV\to 0,\quad \text{for}\ \ N\to\infty. \tag{43}$$

In the thermodynamic limit ($N\to\infty$), Eq. (35) is the same as the FP equation governing the Brownian motion of particles in Wigner-Dyson gas (also the Sutherland model) with particle positions and time in the latter replaced by μ_i and τ in the former. This remains valid also for the case where $Q(x)$ is chosen to be an arbitrary function expandable in a Taylor series. Thus we find that, under a perturbation taken from an ensemble with a sufficiently general distribution, the distribution of eigenvalues of the quantum system evolves in the same way as the distribution of particle position in the Wigner-Dyson gas for arbitrary initial conditions. This also implies that all the moments of the eigenvalues calculated for a parameter value τ will be equal to the corresponding moments of particle positions of the latter. But the motion of eigenvalues is not Brownian because the randomness in their motion comes from the matrix V which acts like quenched disorder [9].

This also implies the equivalence of second-order parametric correlators in the two cases in the thermodynamic limit because they can be expressed as a sum over various moments of eigenvalues, weighted suitably and then averaged over equilibrium initial conditions. However, the FP equation being Markovian in nature, its equivalence for the two cases cannot lead us to a similar conclusion for the higher-order correlators, which involve moments at more than one parameter value and therefore multiple parameter averaging. As mentioned in [9], the n-point correlations for the Wigner-Dyson gas can be expressed in terms of the two-point functions, yielding an n-matrix integral while for RM models of quantum chaos, the n-point function remains a two-matrix integral.

It should be noted here that we have considered the case only for N-independent coefficients in the polynomial; a more careful analysis is required when these have different N dependence. For example, if one or more coefficients are N dependent, increasing with N increasing, the contribution to Eq. (35) from terms in Eqs. (38)–(40) is no longer negligible and therefore the evolution of P is no longer the same as in the Gaussian case.

V. CONCLUSION

In this paper, we have analytically studied the response of energy levels of complex quantum systems to external perturbations modeled by generalized random matrix ensembles. Our results indicate the universality of two-point parametric density correlators as well as the static correlators of all orders, thus agreeing with the numerically observed results for complex systems. One interesting feature revealed by our study is that for both the localized and the delocalized quantum dynamics of these systems, the second-order parametric correlations can be shown to have the same form except that the definition of the effective parameter is different in the two cases. The method adopted here prohibits us from making any statement about universality of the higher-order parametric correlations.

Further, for the reasons given in Secs. I and IV, the correspondence shown between the Wigner-Dyson gas and the quantum Hamiltonian $H=H_0+xV$ with V having a sufficiently general nature also implies the equivalence between the space-time correlators of the Calogero-Sutherland system and the second-order parametric correlations of H. This further strengthens the idea contained in [1] and supported by studies in [9], namely, the 1D Sutherland model provides a model Hamiltonian for the dynamics of eigenvalues of quantum chaotic systems [10]. The equivalence shown between the FP equation governing the evolution of eigenvalues in the GG and SG cases also turns out to be quite helpful in studying the correlations for various difficult but physically significant situations (depending on the variances of the perturbation matrix) in the former case, by using the technique given in Sec. III. Note that our result is quite general as we have shown the equivalence for arbitrary choice of variances. As already mentioned in Sec. II, the results obtained in Secs. II, III, and IV are also valid for the level dynamics of unitary operators U. Note that the analogy between the statistical properties of nonequilibrium circular ensembles that model the eigenvalue spectra of unitary operators and nonequilibrium standard Gaussian ensembles has already been established [12].

Our study still leaves many important questions unanswered. For example, universality or its absence among higher-order correlations of complex systems and their similarities or differences with the Wigner-Dyson gas is not fully understood. Further, the results here have been obtained for explicit averaging over a generalized random perturbing potential. It would be interesting to know whether similar conclusions can also be obtained for the complex systems where ensemble averaging is not valid (e.g., billiards) and eigenvalue statistics should be computed as an average over the N eigenvalues. Intution suggests a positive answer, since sufficiently complex systems are quite likely to be self-averaging. This implies that a single choice of matrices from a particular distribution are representative in the large N limit of the entire distribution and therefore an average over eigenvalues should give the same result as an ensemble average.

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APPENDIX

For H a complex-Hermitian matrix, its eigenvalue equation is given by $\Lambda = UHU^{+}$ with Λ as eigenvalue matrix, and U the eigenvector matrix, which is unitary. Now for a complex Hermitian perturbation $V_{ij} = V_{ij;1} + iV_{ij;2} = V^{*}_{ji}$ (with $V_{ii;2} = 0$), the rate of change of eigenvalues and eigenvectors, with respect to various components of matrix elements of V, can be described as follows:

$$\frac{\partial \lambda_n}{\partial \tau} = -\Omega^2 \lambda_n + \frac{\Omega}{h} \sum_{i \leq j} \frac{1}{g_{ij}} [V_{ij;1}(U_{ni}U^{*}_{nj} + U_{nj}U^{*}_{ni}) + iV_{ij;2}(U_{ni}U^{*}_{nj} - U_{nj}U^{*}_{ni})], \tag{A1}$$

where

$$\frac{\partial H}{\partial \tau} = -\Omega^2 H + \frac{V\Omega}{h}. \tag{A2}$$

Further,

$$\frac{\partial \lambda_n}{\partial V_{kl;1}} = \frac{h(\tau)}{\Omega g_{kl}} [U_{nk}U^{*}_{nl} + U_{nl}U^{*}_{nk}], \tag{A3}$$

$$\frac{\partial \lambda_n}{\partial V_{kl;2}} = \frac{h(\tau)}{\Omega g_{kl}} [U_{nk}U^{*}_{nl} - U_{nl}U^{*}_{nk}] \quad (k \neq l), \tag{A4}$$

$$\frac{\partial \lambda_n}{\partial V_{kl;2}} = 0 \quad (\text{if } k = l), \tag{A5}$$

and

$$\frac{\partial U_{np}}{\partial V_{kl;1}} = \frac{-h(\tau)}{\Omega g_{kl}} \sum_{m \neq n} \frac{1}{\lambda_m - \lambda_n} U_{mp}(U^{*}_{mk}U_{nl} + U^{*}_{ml}U_{nk}), \tag{A6}$$

$$\frac{\partial U_{np}}{\partial V_{kl;2}} = \frac{-h(\tau)}{\Omega g_{kl}} \sum_{m \neq n} \frac{1}{\lambda_m - \lambda_n} U_{mp}(U^{*}_{mk}U_{nl} - U^{*}_{ml}U_{nk}) \tag{A7}$$

and

$$\sum_{k \leq l} \left[\frac{\partial(U_{nk}U^{*}_{nl} + U_{nl}U^{*}_{nk})}{\partial V_{kl;1}} + i \frac{\partial(U_{nk}U^{*}_{nl} - U_{nl}U^{*}_{nk})}{\partial V_{kl;2}} \right] = -\frac{2h(\tau)}{\Omega g_{kl}} \sum_{m \neq n} \frac{1}{\lambda_n - \lambda_m}. \tag{A8}$$

For the real-symmetric case, the corresponding relations can be obtained by using $U_{ij} = U^{*}_{ij}$ (as the eigenvector matrix is now real-orthogonal) in Eqs. (A1)–(A8) and taking $V_{ij;2} = 0$ for all values of i,j (also found in Ref. [9]).

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